SOLUTIONS

UBC Math104/184 Exam (December 2009)

1. (a) $\lim_{x\to\infty}\dfrac{4x^2+x+a}{ax^2-7x} = \lim_{x\to\infty}\dfrac{\frac{4x^2}{x^2}+\frac{x}{x^2}+\frac{a}{x^2}}{\frac{ax^2}{x^2}-\frac{7x}{x^2}} = \lim_{x\to\infty}\dfrac{4+\frac{1}{x}+\frac{a}{x^2}}{a-\frac{7}{x}} = \dfrac{4+0+0}{a-0} = \dfrac{4}{a} = 2$, so $a=2$.

(b) $\lim_{x\to4}\dfrac{\sqrt{2x-1}-\sqrt{7}}{x-4} = \lim_{x\to4}\dfrac{\sqrt{2x-1}-\sqrt{7}}{x-4}\cdot\dfrac{\sqrt{2x-1}+\sqrt{7}}{\sqrt{2x-1}+\sqrt{7}} = \lim_{x\to4}\dfrac{(2x-1)-7}{(x-4)(\sqrt{2x-1}+\sqrt{7})}$

$$= \lim_{x\to4}\frac{2x-8}{(x-4)(\sqrt{2x-1}+\sqrt{7})} = \lim_{x\to4}\frac{2(x-4)}{(x-4)(\sqrt{2x-1}+\sqrt{7})}$$

$$= \lim_{x\to4}\frac{2}{\sqrt{2x-1}+\sqrt{7}} = \frac{2}{\sqrt{7}+\sqrt{7}} = \frac{2}{2\sqrt{7}} = \frac{1}{\sqrt{7}}.$$

(c) $\lim_{x\to0^-}f(x) = \lim_{x\to0^-}\dfrac{x-c}{c+1} = \dfrac{-c}{c+1}$, and $\lim_{x\to0^+}f(x) = \lim_{x\to0^+}(x^2+c) = c$.

Therefore $\lim_{x\to0}f(x)$ only exists if $\lim_{x\to0^-}f(x)=\lim_{x\to0^+}f(x)$, i.e. if $\dfrac{-c}{c+1}=c$. Therefore,

$-c=c(c+1) \;\Rightarrow\; c(c+1)+c=0 \;\Rightarrow\; c(c+2)=0$ so either $c=0$ or $c=-2$.

(d) $R=pq=p(7-2\ln p)$ so marginal revenue is

$$\frac{dR}{dp} = p\left(0-2\cdot\frac{1}{p}\right)+(7-2\ln p)\cdot 1 = -2+(7-2\ln p) = 5-2\ln p.$$

(e) $y'=\dfrac{(1+x^2)\cdot1-x\cdot2x}{(1+x^2)^2} = \dfrac{1+x^2-2x^2}{(1+x^2)^2} = \dfrac{1-x^2}{(1+x^2)^2}$.

At a critical point, $y'=0$ so $1-x^2=0$. Therefore $1=x^2$ so $x=\pm1$.

$$y''=\frac{(1+x^2)^2\cdot(-2x)-(1-x^2)\cdot[2(1+x^2)\cdot2x]}{(1+x^2)^4} = \frac{(1+x^2)\cdot(-2x)-(1-x^2)\cdot4x}{(1+x^2)^3}$$

$$= \frac{2x[-(1+x^2)-2(1-x^2)]}{(1+x^2)^3} = \frac{2x[-1-x^2-2+2x^2]}{(1+x^2)^3} = \frac{2x(x^2-3)}{(1+x^2)^3}.$$

At $x=-1$, $y''=\dfrac{-2\cdot(-2)}{(1+1)^3}=\dfrac{1}{2}>0$, so this is a local minimum.

At $x=1$, $y''=\dfrac{2\cdot(-2)}{(1+1)^3}=-\dfrac{1}{2}<0$, so this is a local maximum.

When $x=1$, $y=\dfrac{1}{1+1^2}=\dfrac{1}{2}$, so the local maximum is $(x,y)=(1,\frac{1}{2})$.

(f) $f'(x) = 12x^3 - 12x^2 - 12x + 12 = 12(x^3 - x^2 - x + 1) = 12[(x^3 - x^2) - (x-1)] = 12[x^2(x-1) - (x-1)]$

$$= 12(x^2 - 1)(x-1) = 12(x+1)(x-1)(x-1) = 12(x+1)(x-1)^2.$$

At a critical point, $f'(x) = 0$ so either $x = 1$ or $x = -1$. In the interval $[0,2]$, there is only the critical point $x = 1$, and $f(1) = 3 - 4 - 6 + 12 + 1 = 6$.

At the endpoints, $f(0) = 1$, and $f(2) = 3 \cdot 16 - 4 \cdot 8 - 6 \cdot 4 + 24 + 1 = 17$. So the absolute minimum occurs at $x = 0$.

(g) $\dfrac{d}{dx}(y^2) = \dfrac{d}{dx}(x^3 + 1) \;\Rightarrow\; 2yy' = 3x^2 \;\Rightarrow\; y' = \dfrac{3x^2}{2y}$.

The slope of the tangent line at the point $(2,-3)$ is $m = y'\big|_{(2,-3)} = \dfrac{3 \cdot 2^2}{2 \cdot (-3)} = \dfrac{12}{-6} = -2$.

Since the tangent line passes through the point $(2,-3)$, its equation is $y + 3 = -2(x-2)$.

The y-intercept occurs when $x = 0$ giving $y + 3 = -2(0-2) = 4$ or $y = 1$, so the y-intercept is $(0,1)$.

(h) By the Chain Rule, $g'(x) = f'(e^x) \cdot \dfrac{d}{dx}(e^x) = \dfrac{e^x - 1}{e^x + 1} \cdot e^x$.

(i) Let $f(x) = \sqrt[3]{x} = x^{1/3}$, and $a = 27$. Then $f'(x) = \frac{1}{3}x^{-2/3} = \dfrac{1}{3(\sqrt[3]{x})^2}$. The linear approximation to $f(x)$ is

$$f(x) \approx f(a) + f'(a)(x-a) = f(27) + f'(27)(x-27) = \sqrt[3]{27} + \frac{1}{3(\sqrt[3]{27})^2}(x-27),$$

or $\quad \sqrt[3]{x} \approx 3 + \frac{1}{27}(x-27)$. Plugging in $x = 28$, gives $\sqrt[3]{28} \approx 3 + \frac{1}{27}(28-27) = 3\frac{1}{27} = \frac{82}{27} = 3.037037$.

The calculator value is $\sqrt[3]{28} = 3.036589$. The absolute value of the error is $\left|\sqrt[3]{28} - \frac{82}{27}\right| = 0.000448$.

(j) Since revenue is $R = pq$, therefore $\dfrac{dR}{dt} = p\dfrac{dq}{dt} + q\dfrac{dp}{dt}$. Plugging in $p = \ln 2$, $q = 1$ and $\dfrac{dR}{dt} = \dfrac{1}{e}$, gives $\dfrac{1}{e} = \ln 2\dfrac{dq}{dt} + 1\dfrac{dp}{dt}$. So $\ln 2\dfrac{dq}{dt} = \dfrac{1}{e} - \dfrac{dp}{dt}$ or $\dfrac{dq}{dt} = \dfrac{1}{\ln 2}\left(\dfrac{1}{e} - \dfrac{dp}{dt}\right)$. Next,

$$\frac{d}{dt}(e^{pq} - q^2) = \frac{d}{dt}(1) \;\Rightarrow\; e^{pq}\left(p\frac{dq}{dt} + q\frac{dp}{dt}\right) - 2q\frac{dq}{dt} = 0 \;\Rightarrow\; e^{pq}\frac{dR}{dt} - 2q \cdot \left[\frac{1}{\ln 2}\left(\frac{1}{e} - \frac{dp}{dt}\right)\right] = 0.$$

So $e^{(\ln 2)\cdot 1}\left(\dfrac{1}{e}\right) - 2 \cdot 1 \cdot \left[\dfrac{1}{\ln 2}\left(\dfrac{1}{e} - \dfrac{dp}{dt}\right)\right] = 0 \;\Rightarrow\; \dfrac{2}{e} - \left[\dfrac{2}{\ln 2}\left(\dfrac{1}{e} - \dfrac{dp}{dt}\right)\right] = 0 \;\Rightarrow\; \dfrac{2}{e} = \dfrac{2}{\ln 2}\left(\dfrac{1}{e} - \dfrac{dp}{dt}\right)$.

Therefore $\dfrac{\ln 2}{e} = \dfrac{1}{e} - \dfrac{dp}{dt}$ or $\dfrac{dp}{dt} = \dfrac{1 - \ln 2}{e}$.

(k) After t years, investment A is worth $A = Pe^{rt} = 70{,}200e^{0.13t}$ and investment B is worth $B = Pe^{rt} = 60{,}000e^{0.14t}$. These investments will have the same value when $A = B$, i.e. when $70{,}200e^{0.13t} = 60{,}000e^{0.14t}$. Therefore $\dfrac{70{,}200}{60{,}000} = \dfrac{e^{0.14t}}{e^{0.13t}}$ or $\dfrac{351}{300} = e^{0.01t}$. Therefore, $0.01t = \ln\frac{351}{300}$ so $t = 100\ln\frac{351}{300} \approx 15.70$ years.

(l) $f'(x) = 2^{\sin x}\ln 2 \cdot \cos x$.

(m) $f(x) = \frac{9}{14}x^{1/3}(x^2 - 7) = \frac{9}{14}x^{7/3} - \frac{9}{2}x^{1/3}$; $\qquad f'(x) = \frac{3}{2}x^{4/3} - \frac{3}{2}x^{-2/3}$;

$$f''(x) = 2x^{1/3} + x^{-2/3} = 2x^{1/3} + \frac{1}{x^{2/3}} = \frac{2x^{1/3}x^{2/3}}{x^{2/3}} + \frac{1}{x^{2/3}} = \frac{2x+1}{(\sqrt[3]{x})^2}.$$

$f(x)$ is concave up when $f''(x) = \dfrac{2x+1}{(\sqrt[3]{x})^2} > 0$, which occurs when the numerator $2x+1$ is positive (since the denominator $(\sqrt[3]{x})^2$ is always positive), i.e. when $2x+1>0$, which is on the interval $(-\frac{1}{2},\infty)$.

(n) $f(x) = \cos(\pi - 5x)$; $\qquad f(0) = \cos(\pi) = -1$;

$f'(x) = -\sin(\pi - 5x)\cdot(-5) = 5\sin(\pi - 5x)$; $\qquad f'(0) = 5\sin(\pi) = 0$;

$f''(x) = 5\cos(\pi - 5x)\cdot(-5) = -25\cos(\pi - 5x)$; $\qquad f''(0) = -25\cos(\pi) = 25$;

$f'''(x) = 25\sin(\pi - 5x)\cdot(-5) = -125\sin(\pi - 5x)$; $\qquad f'''(0) = -125\sin(\pi) = 0$.

The third degree Taylor polynomial of $f(x)$ is

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = -1 + 0x + \frac{25}{2!}x^2 + \frac{0}{3!}x^3 = -1 + \tfrac{25}{2}x^2.$$

2. (a) $$f'(x) = \frac{(x^2+1)\cdot 2(x+1) - (x+1)^2\cdot 2x}{(x^2+1)^2} = \frac{2(x+1)\cdot[(x^2+1) - x(x+1)]}{(x^2+1)^2} = \frac{2(x+1)\cdot[(x^2+1)-(x^2+x)]}{(x^2+1)^2}$$

$$= \frac{2(1+x)(1-x)}{(x^2+1)^2} = \frac{2(1-x^2)}{(x^2+1)^2}.$$

At a critical point $f'(x) = 0$, so $\dfrac{2(1+x)(1-x)}{(x^2+1)^2} = 0$. Therefore $x = 1$ or $x = -1$.

So there are two critical points, $(1, f(1)) = (1,2)$ and $(-1, f(-1)) = (-1,0)$.

If $x < -1$, then $f'(x) = \dfrac{2(1+x)(1-x)}{(x^2+1)^2} = \dfrac{2(\text{negative})(\text{positive})}{(\text{positive})} = \text{negative}$, so $f(x)$ is decreasing.

If $-1 < x < 1$, then $f'(x) = \dfrac{2(1+x)(1-x)}{(x^2+1)^2} = \dfrac{2(\text{positive})(\text{positive})}{(\text{positive})} = \text{positive}$, so $f(x)$ is increasing.

If $x>1$, then $f'(x)=\dfrac{2(1+x)(1-x)}{(x^2+1)^2} = \dfrac{2\,(\text{positive})(\text{negative})}{(\text{positive})}$ = negative, so $f(x)$ is decreasing.

So $f(x)$ is decreasing on the intervals $(-\infty,-1)$ and $(1,\infty)$, and increasing on the interval $(1,\infty)$.

Therefore $(-1,0)$ is a local minimum and $(1,2)$ is a local maximum.

(b) $$f''(x)=\frac{(x^2+1)^2\cdot(-4x)-2(1-x^2)\cdot[2(x^2+1)\cdot 2x]}{(x^2+1)^4} = \frac{-4x(x^2+1)[(x^2+1)+2(1-x^2)]}{(x^2+1)^4}$$
$$= \frac{-4x[(x^2+1)+(2-2x^2)]}{(x^2+1)^3} = \frac{-4x(3-x^2)}{(x^2+1)^3} = \frac{4x(x^2-3)}{(x^2+1)^3}.$$

At an inflection point $f''(x)=0$, so $\dfrac{4x(x^2-3)}{(x^2+1)^3}=0$. Therefore $4x=0$ or $x^2-3=0$. So there are three possible inflection points, namely $x=0$, $x=\sqrt{3}$ and $x=-\sqrt{3}$.

If $x<-\sqrt{3}$, then $f''(x)=\dfrac{4x(x^2-3)}{(x^2+1)^3} = \dfrac{(\text{negative})(\text{positive})}{(\text{positive})}$ = negative, so $f(x)$ is concave down.

If $-\sqrt{3}<x<0$, then $f''(x)=\dfrac{4x(x^2-3)}{(x^2+1)^3} = \dfrac{(\text{negative})(\text{negative})}{(\text{positive})}$ = positive, so $f(x)$ is concave up.

If $0<x<\sqrt{3}$, then $f''(x)=\dfrac{4x(x^2-3)}{(x^2+1)^3} = \dfrac{(\text{positive})(\text{negative})}{(\text{positive})}$ = negative, so $f(x)$ is concave down.

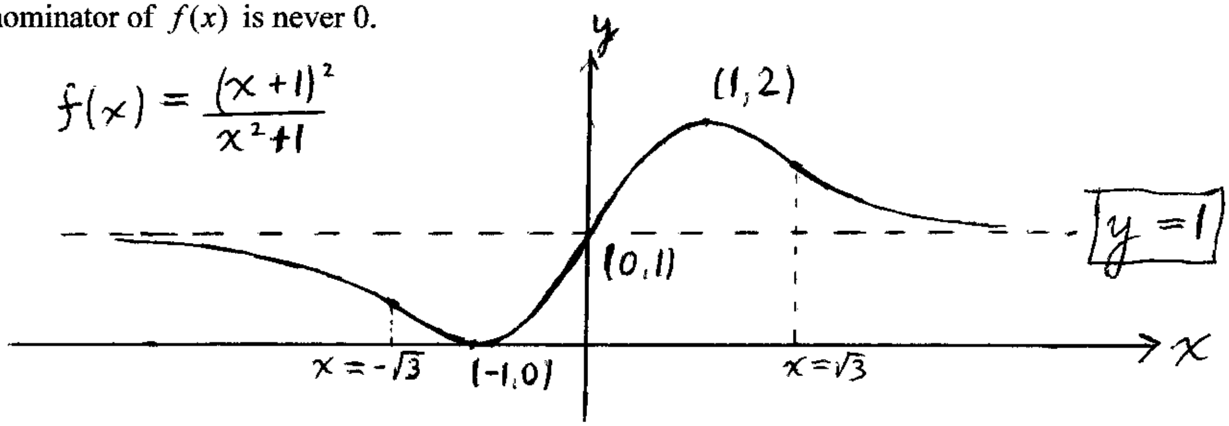
If $x>\sqrt{3}$, then $f''(x)=\dfrac{4x(x^2-3)}{(x^2+1)^3} = \dfrac{(\text{positive})(\text{positive})}{(\text{positive})}$ = positive, so $f(x)$ is concave up.

So $f(x)$ is concave down on the intervals $(-\infty,-\sqrt{3})$ and $(0,\sqrt{3})$, and concave up on the intervals $(-\sqrt{3},0)$ and $(\sqrt{3},\infty)$. So there really are inflection points at $x=0$, $x=\sqrt{3}$ and $x=-\sqrt{3}$ (since the concavity does change at these points).

(c) Since $f(x)=\dfrac{(x+1)^2}{x^2+1} = \dfrac{x^2+2x+1}{x^2+1} = \dfrac{\frac{x^2}{x^2}+\frac{2x}{x^2}+\frac{1}{x^2}}{\frac{x^2}{x^2}+\frac{1}{x^2}} = \dfrac{1+\frac{2}{x}+\frac{1}{x^2}}{1+\frac{1}{x^2}}$, therefore $\lim\limits_{x\to\pm\infty} f(x)=\dfrac{1+0+0}{1+0}=1$.

So $y=1$ is a horizontal asymptote for $y=f(x)$. There are no vertical asymptotes since the denominator of $f(x)$ is never 0.

(d)



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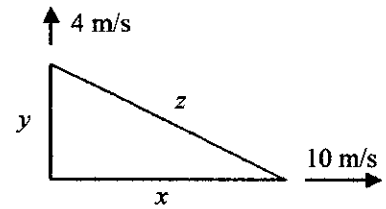

3. Let y be the altitude of the balloon, x the distance the cyclist has travelled since passing beneath the balloon, and z the distance between the cyclist and the balloon. Then $z^2 = x^2 + y^2$. Therefore

$$\frac{d}{dt}(z^2) = \frac{d}{dt}(x^2+y^2) \Rightarrow 2z\frac{dz}{dt} = 2x\frac{dx}{dt}+2y\frac{dy}{dt}$$

or
$$z\frac{dz}{dt} = x\frac{dx}{dt}+y\frac{dy}{dt}.$$

5 seconds after the cyclist passes beneath the balloon, $x=5\cdot 10=50$, $y=40+5\cdot 4=60$,

$z=\sqrt{x^2+y^2} = \sqrt{50^2+60^2} = \sqrt{2500+3600} = \sqrt{6100} = 10\sqrt{61}$, $\frac{dx}{dt}=10$, and $\frac{dy}{dt}=4$.

Plugging these values into the above equation gives

$$10\sqrt{61}\cdot\frac{dz}{dt} = 50\cdot 10+60\cdot 4 = 740 \text{ or } \frac{dz}{dt} = \frac{740}{10\sqrt{61}} = \frac{74}{\sqrt{61}} \text{ m/s}.$$

4. (a) Since $\ln q - 2\ln p + 0.02p = 7$, therefore $\ln q = 7+2\ln p - 0.02p$, so

$$q = e^{7+2\ln p-0.02p} = e^7 e^{2\ln p}e^{-0.02p} = e^7(e^{\ln p})^2 e^{-0.02p}.$$

So $q=f(p)=e^7p^2e^{-0.02p}$, and

$$f'(p)=e^7[p^2\cdot(e^{-0.02p}\cdot(-0.02))+e^{-0.02p}\cdot 2p] = e^7e^{-0.02p}[2p-0.02p^2] = e^7e^{-0.02p}p[2-0.02p].$$

The elasticity of demand is given by

$$E(p) = -\frac{pf'(p)}{f(p)} = -\frac{p\cdot e^7e^{-0.02p}p[2-0.02p]}{e^7p^2e^{-0.02p}} = -\frac{p^2\cdot[2-0.02p]}{p^2} = -(2-0.02p) = 0.02p-2.$$

When $p=200$, $E(200)=0.02\cdot(200)-2=4-2=2>1$. Since the elasticity is greater than 1, the revenue will decrease if p increases and increase if p decreases. Therefore revenue will increase if the price is lowered slightly.

(b) Revenue is $R=pq=pf(p)=p\cdot e^7p^2e^{-0.02p}=e^7p^3e^{-0.02p}$. Therefore

$$\frac{dR}{dp}=e^7[p^3\cdot(e^{-0.02p}\cdot(-0.02))+e^{-0.02p}\cdot 3p^2]=e^7e^{-0.02p}[3p^2-0.02p^3]=e^7p^2e^{-0.02p}[3-0.02p].$$

At a critical point $\frac{dR}{dp}=0$, so $3-0.02p=0$ or $p=\frac{3}{0.02}=150$.

When $p\le 150$, $\frac{dR}{dp}=e^7p^2e^{-0.02p}[3-0.02p]\ge 0$ so R is increasing, and when $p\ge 150$, $\frac{dR}{dp}=e^7p^2e^{-0.02p}[3-0.02p]\le 0$ so R is decreasing. Therefore, the maximum value of the revenue occurs when $p=150$.

5. Let x be the number of orders made during the year. Since 10,000 motorcycles will be sold during the year, each order is for $N = \dfrac{10{,}000}{x}$ motorcycles. The average number of motorcycles in inventory throughout the year is half of that, i.e. $\frac{1}{2}N = \dfrac{5{,}000}{x}$. The total cost is therefore

$$C(x) = 10{,}000x + 200 \cdot (\tfrac{1}{2}N) = 10{,}000x + 200 \cdot \frac{5{,}000}{x} = 10{,}000x + \frac{1{,}000{,}000}{x} = 10{,}000(x + 100x^{-1}).$$

At a critical point $C'(x) = 10{,}000(1 - 100x^{-2}) = 0$, so $1 = 100x^{-2} = \dfrac{100}{x^2}$ or $x^2 = 100$.

Therefore $x = 10$ (since $x > 0$). So they should make 10 orders per year, and each order should be for $N = \dfrac{10{,}000}{10} = 1000$ motorcycles.

6. Setting $P = 100{,}050$, $R = 900$, and $N = 20 \cdot 12 = 240$, gives

$$100{,}050i + 900\,[(1+i)^{-240} - 1] = 0.$$

Let $f(x) = 100{,}050x + 900\,[(1+x)^{-240} - 1]$.

Then $f'(x) = 100{,}050 + 900\,[-240(1+x)^{-241} - 0] = 100{,}050 - 216{,}000(1+x)^{-241}$.

Starting with the initial guess $x_0 = 0.02$ gives

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.02 - \frac{f(0.02)}{f'(0.02)} = 0.02 - \frac{100{,}050 \cdot (0.02) + 900\,[(1.02)^{-240} - 1]}{100{,}050 - 216{,}000(1.02)^{-241}}$$

$$= 0.02 - \frac{1108.76607}{98222.68947} = 0.0087117114$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.0087117114 - \frac{f(0.0087117114)}{f'(0.0087117114)}$$

$$= 0.0087117114 - \frac{100{,}050 \cdot (0.0087117114) + 900\,[(1.0087117114)^{-240} - 1]}{100{,}050 - 216{,}000(1.0087117114)^{-241}}$$

$$= 0.0087117114 - \frac{83.84664}{73345.06339} = 0.0075685309.$$

The monthly rate of interest is approximately 0.756%.